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Strength modeling using Weibull distributions

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Abstract

The two-parameter Weibull distribution is the most popular model for material strength. However, it may not be a good model for all materials over a wide range of sizes. In this note, a comprehensive review of the known variations of the two-parameter Weibull distribution is provided to help providing better modeling. Over 20 variations are reviewed. The appropriateness of the variations is discussed as models for brittle versus ductile strength. A comparison study of a selection of the variations is also provided. It is hoped that this review will also serve as an important reference and encourage developments of further variations of the two-parameter Weibull distribution.

Keywords: Material strength; Statistical modeling; Weibull distribution

1. Introduction

The most widely used distribution for characterizing material strength is the two parameter Weibull distribution. If X is a random variable representing some material strength then the probability distribution function (pdf) and the cumulative distribution function (cdf) of the Weibull distribution can be expressed as

$$p(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp\{-(\lambda x)^{\alpha}\} \tag{1}$$

and

$$P(x) = 1-\exp\{-(\lambda x)^{\alpha}\}, \tag{2}$$

respectively, for $x > 0$, $\lambda > 0$ and $\alpha > 0$. The parameters λ and α are referred to as the scale and shape parameters, respectively. The Rayleigh and Levy distributions are the particular cases of (1)-(2) for α = 2 and α = 1/2, respectively. The *n*th moment of X associated with (1)-(2) is $E(X^n) = \lambda^{-n}\Gamma((\alpha+n)/\alpha)$.

While (1)-(2) has been shown to be a good approximation for the distribution of average strength of many brittle materials, it may not be a good representation for all materials over a wide range of sizes. This phenomenon is especially significant because it might result in a serious strength overestimation or underestimation. This problem could be remedied by using one of the many generalizations of (1)-(2) that have been proposed in the statistical literature. The aim of this note is to help this by providing a comprehensive review of the known variants of (1)-(2). Over 20 variations of (1)-(2) are reviewed-formulas for the distribution functions and the moments are given for each. Apart from helping to provide better modeling, it is hoped that this review will serve as an important reference and encourage developments of further generalizations of (1)-(2).

We also discuss the appropriateness of the variations as models for brittle versus ductile strength and provide a comparison study of a selection of the variations using six data sets on fracture toughness.

2. Reflected Weibull distribution

If X has the Weibull distribution given by (1) then -X is said to have the reflected Weibull distribu-

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tion. The pdf and the cdf are given by

$$p(x) = \lambda\alpha(-\lambda x)^{\alpha-1} \exp\{-(-\lambda x)^{\alpha}\} \quad (3)$$

and

$$P(x) = \exp\{-(-\lambda x)^{\alpha}\} \quad (4)$$

respectively, for $-\infty < x < 0$, $\lambda > 0$ and $\alpha > 0$. The associated mean and the variance are $-\lambda^{-1}\Gamma(1+1/\alpha)$ and $\lambda^{-2}\{\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)\}$, respectively. For further details of this distribution see Cohen [1].

3. Double Weibull distribution

A distribution made up by piecing (1) and (3) together is the double Weibull distribution [2]. The pdf and the cdf are given by

$$p(x) = (\lambda\alpha/2)|\lambda x|^{\alpha-1} \exp\{-|\lambda x|^{\alpha}\} \quad (5)$$

and

$$P(x) = \begin{cases} (1/2)\exp\{-|\lambda x|\alpha\}, & \text{if } x < 0, \\ 1-(1/2)\exp\{-|\lambda x|\alpha\}, & \text{if } x \geq 0, \end{cases} \quad (6)$$

respectively, for $-\infty < x < \infty$, $\alpha > 0$ and $\lambda > 0$. The nth moment corresponding to (5) is:

$$E(X^n) = \begin{cases} \lambda^{-n}\Gamma((\alpha+n)/\alpha, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}. \end{cases}$$

4. Log Weibull distribution

If X has theWeibull distribution given by (1) then log$X$ is said to have the logWeibull distribution. The pdf and the cdf take the forms

$$p(x) = \frac{1}{b}\exp\left(\frac{x-a}{b}\right)\exp\left\{-\exp\left(\frac{x-a}{b}\right)\right\} \quad (7)$$

and

$$P(x) = 1-\exp\left\{-\exp\left(\frac{x-a}{b}\right)\right\}, \quad (8)$$

respectively, for $-\infty < x < \infty$, $-\infty < a < \infty$, and $b > 0$. The mean and the variance of this distribution are $a - \gamma b$ and $\pi^2 b^2/6$, respectively, where γ denotes the Euler's constant. For further details see White [3].

5. Inverse Weibull distribution

If X has the Weibull distribution given by (1) then $1/X$ is said to have the inverse Weibull distribution. The pdf and the cdf take the forms

$$p(x) = \alpha\lambda^{\alpha} x^{-\alpha-1} \exp\{-(x/\lambda)^{-\alpha}\} \quad (9)$$

and

$$P(x) = \exp\{-(x/\lambda)^{-\alpha}\}, \quad (10)$$

respectively, for $x > 0$, $\alpha > 0$ and $\lambda > 0$. The nth moment corresponding to (9) is:

$$E(X^n) = \lambda^n \Gamma((\alpha-n)/\alpha).$$

This distribution is also referred to as the complementary or the reciprocal Weibull distribution. For further details see Drapella [4] and Mudholkar and Kollia [5].

6. Truncated Weibull distribution

A doubly truncated version of (1) is given by the pdf and the cdf

$$q(x) = \frac{p(x)}{P(b)-P(a)} \quad (11)$$

and

$$Q(x) = \frac{P(x)-P(a)}{P(b)-P(a)} \quad (12)$$

respectively, for $0 \leq a < x < b < \infty$, where $p(x)$ and $P(x)$ are given by (1) and (2), respectively. The associated nth moment is given by

$$E(X^n) = \frac{\exp\{(\lambda b)^{\alpha}\}}{1-\exp\{-(\lambda a)^{\alpha}\}}\lambda^{-n}\Gamma\left(\frac{n}{\alpha}+1, (\lambda b)^{\alpha}\right) - \Gamma\left(\frac{n}{\alpha}+1, (\lambda a)^{\alpha}\right),$$

where $\Gamma(\cdot,\cdot)$ is the complementary incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1}\exp(-t)dt.$$

For further details about this distribution see McEwen and Parresol [6].

7. Marshall and Olkin's Weibull distribution

Marshall and Olkin [7] proposed an extended Weibull distribution given by the pdf

$$p(x) = \frac{\nu\lambda\alpha(\lambda x)^{\alpha-1}\exp\{-(\lambda x)^{\alpha}\}}{1-(1-\nu)\exp\{-(\lambda x)^{\alpha}\}} \quad (13)$$

for $x > 0$, $\nu > 0$, $\lambda > 0$ and $\alpha > 0$. The Weibull distribution in (1) is the particular case of (13) for

$\nu = 1$. The cdf and the nth moment associated with (13) are:

$$P(x) = 1 - \frac{\nu \exp\{-(\lambda x)^\alpha\}}{1-(1-\nu)\exp\{-(\lambda x)^\alpha\}}$$

and

$$E(X^n) = \frac{n\lambda}{\alpha}\sum_{k=0}^{\infty}\frac{(1-\nu)^k}{(k+1)^{n/\alpha}}\Gamma\left(\frac{n}{\alpha}\right),$$

respectively, with the latter being valid for $|1-\nu| \le 1$.

8. Pseudo Weibull distribution

The pseudo Weibull distribution due to Voda [8] is given by the pdf

$$p(x) = \alpha x^\alpha \lambda^{\alpha+1}\{\Gamma(1+1/\alpha)\}^{-1}\exp\{-(\lambda x)^\alpha\} \quad (14)$$

for $x>0, \alpha>0$ and $\lambda>0$. Note that this form is obtained by multiplying (1) by an additional x term. The cdf and the nth moment corresponding to (14) are :

$$P(x) = \frac{\gamma(1+1/\alpha, (\lambda x)^\alpha)}{\Gamma(1+1/\alpha)}$$

and

$$E(X^n) = \frac{\Gamma(1+(1+n)/\alpha)}{\lambda^{n/\alpha}\Gamma(1+1/\alpha)},$$

respectively, where $\gamma(\cdot,\cdot)$ denotes the incomplete gamma function defined by

$$\gamma(a,x) = \int_0^x t^{a-1}\exp(-t)dt.$$

9. Stacy's Weibull distribution

Stacy [9] proposed a distribution with the pdf

$$p(x) = c\beta^{-c\alpha}\{\Gamma(\alpha)\}^{-1}x^{c\alpha-1}\exp\{-(x/\beta)^c\} \quad (15)$$

for $x>0, c>0, \alpha>0$ and $\beta>0$. The Weibull distribution in (1) arises as the particular case of (15) for $\alpha=1$. The cdf and the nth moment corresponding to (15) are:

$$P(x) = \{\Gamma(\alpha)\}^{-1}\gamma\left(\alpha, \left(\frac{x}{\beta}\right)^c\right)$$

and

$$E(X^n) = \beta^n\Gamma(\alpha+n/c)/\Gamma(\alpha),$$

respectively. A further extension of (15) due to Ghitany [10] has the pdf given by

$$p(x) = \frac{\alpha^{(m-1)/\beta+1-\lambda}\beta x^{m-\beta-2}\left(n+x\beta\right)^{-\lambda}}{\Gamma_\lambda((m-1)/\beta, \alpha n)}\exp\left\{-\left(\frac{x}{\alpha}\right)^\beta\right\}$$

for $x>0, \alpha>0, \beta>0, \lambda>0, m>0$ and $n>0$, where $\Gamma_\lambda(\cdot,\cdot)$ is defined by

$$\Gamma_\lambda(a,x) = \int_x^\infty \frac{y^{a-1}\exp(-y)}{(y+x)^\lambda}dy.$$

10. Exponentiated Weibull distribution

Mudholkar et al. [11] introduced the exponentiated Weibull distribution given by the pdf and the cdf

$$p(x) = a\alpha\lambda^\alpha x^{\alpha-1}\exp\{-(\lambda x)^\alpha\}[1-\exp\{-(\lambda x)^\alpha\}]^{a-1} \quad (16)$$

and

$$P(x) = [1-\exp\{-(\lambda x)\alpha\}]^a, \quad (17)$$

respectively, for $x>0$, $a>0$, $\alpha>0$ and $\lambda>0$. The Weibull distribution in (1) is the particular case of (16) for $a=1$. The nth moment associated with (16) is given by

$$E(X^n) = a\lambda^{-n} - n\Gamma\left(\frac{n}{\alpha}+1\right)\sum_{i=0}^{\infty}\frac{(1-a)_i}{i!(i+1)^{(n+\alpha)/\alpha}},$$

which can be reduced to the simpler form

$$E(X^n) = \alpha\lambda^{-n}\Gamma\left(\frac{n}{\alpha}+1\right)\sum_{i=0}^{a-1}\frac{(1-a)_i}{i!(i+1)^{(n+\alpha)/\alpha}}$$

if $a \ge 1$ is an integer. For further details of this distribution see Mudholkar and Hutson [12], Nassar and Eissa [13] and Nadarajah and Gupta [14].

11. Xie et al.'s weibull distribution

Xie et al. [15] proposed a modification of the Weibull distribution given by the pdf and the cdf

$$p(x) = \lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1}\exp\left[\left(\frac{x}{\alpha}\right)^\beta + \lambda\alpha\left\{1-\exp\left(\frac{x}{\alpha}\right)^\beta\right\}\right] \quad (18)$$

and

$$P(x) = 1-\exp\left[\lambda\alpha\left\{1-\exp\left(\frac{x}{\varepsilon}\right)^\beta\right\}\right] \quad (19)$$

for $x>0, \lambda>0, \alpha>0$ and $\beta>0$. The Weibull distribution in (1) arises as the limiting case of (18) as $\alpha \to \infty$. Expressions for the moments associated with (18) cannot be obtained in closed form. However, if $k/\beta = n$ is an integer then the kth moment can be expressed as

$$E(X^k) = n\alpha^k \exp(\lambda\alpha) \frac{\partial^{n-1} (\lambda\alpha)^{-\nu} \Gamma(\nu, \lambda\alpha)}{\partial \nu^{n-1}} \quad (20)$$

for $n = 1, 2, \ldots$, where the derivative is evaluated as $\nu \to 0$. For further details of this distribution see Chen [16], Tang et al. [17], Wu et al. [18] and Nadarajah [19].

12. Lai et al.'s Weibull distribution

Another modification of the Weibull distribution developed by Lai et al. [20] is given by the pdf and the cdf

$$p(x) = \lambda(\beta + \nu x) x^{\beta-1} \exp(\nu x) \exp\{-\lambda x^{\beta} \exp(\nu x\} \quad (21)$$

and

$$P(x) = 1 - \exp\{-\lambda x^{\beta} \exp(\nu x)\}, \quad (22)$$

respectively, for $x > 0, \lambda > 0, \beta > 0$ and $\nu > 0$. The Weibull distribution in (1) is the particular case of (21) for $\nu = 0$. The log-Webiull distribution in (7) is the particular case of (21) for $\beta = 0$.

13. Generalized Weibull distributions

Mudholkar and Srivastava [21] and Mudholkar et al. [11] developed three different generalized Weibull distributions. The first of these has the pdf and the cdf given by

$$p(x) = \frac{\lambda\beta(\lambda x)^{\beta-1}}{1 - \nu(\lambda x)^{\beta}} \left[1 - \nu(\lambda x)^{\beta}\right]^{1/\nu} \quad (23)$$

and

$$P(x) = 1 - \left[1 - \nu(\lambda x)^{\beta}\right]^{1/\nu} \quad (24)$$

respectively, for $0 < x < \infty$ (if $\nu \le 0$), $0 < x < \lambda^{-1} \nu^{-1/\beta}$ (if $\nu > 0$), $\lambda > 0$ and $\beta > 0$. The Weibull distribution in (1) arises as the limiting case of (23) for $\nu \to 0$. The *n*th moment associated with (23) is:

$$E(X^n) = \nu^{-n/\beta-1} B\ (1/\nu,\ n/\beta + 1)\ .$$

The second generalization is given by the cdf

$$P(x) = 1 - \left[1 - \nu(1 + x/\beta)^{\beta}\right]^{1/\nu} \quad (25)$$

for $-\infty < x < -\beta$ (if $\beta < 0$ and $\nu < 0$), $-\infty < x < \beta\nu^{-1/\beta} - \beta$ (if $\beta < 0$ and $\nu > 0$), $-\beta < x < \infty$ (if $\beta > 0$ and $\nu < 0$), and $-\beta < x < \beta\nu^{1/\beta} - \beta$ (if $\beta > 0$ and $\nu > 0$). The inverse Weibull (see Section 5) is a particular case of this distribution. The *n*th moment associated with (25) is:

$$E(X^n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \beta^n \nu^{-n/\beta-1} B(1/\nu, n/\beta + 1).$$

The third generalization discussed in Mudholkar and Srivastava [21] and Mudholkar et al. [11] is given by the cdf

$$P(x) = \begin{cases} 1 - \exp\left\{-(1 + x/\beta)^{\beta}\right\}, & \text{if } \beta \ne 0, \\ 1 - \exp\{-\exp(x)\}, & \text{if } \beta = 0 \end{cases} \quad (26)$$

for $-\infty < x < -\beta$ (if $\beta < 0$), $-\beta < x < \infty$ (if $\beta > 0$) and $-\infty < x < \infty$ (if $\beta = 0$). For further details about these generalizations see Mudholkar and Kollia [5].

14. Four and five-parameter weibull distributions

Phani [22] developed a four-parameter Weibull distribution given by the cdf

$$P(x) = 1 - \exp\left\{-\lambda[(x - a)/(b - x)]^{\beta}\right\} \quad (27)$$

for $0 \le a < x < b < \infty, \lambda > 0$ and $\beta > 0$. A five-parameter extension of this given by Kies [23] has the cdf

$$P(x) = 1 - \exp\left\{-\lambda(x - a)^{\beta_1} / b(-x)^{\beta_2}\right\} \quad (28)$$

for $0 \le a < x < b < \infty, \lambda > 0, \beta_1 > 0$ and $\beta_2 > 0$.

15. Mixture Weibull distributions

An additive twofold Weibull mixture distribution can be expressed by the pdf

$$p(x) = \alpha p_1(x) + (1 - \alpha) p_2(x) \quad (29)$$

for $0 < \alpha < 1$, where p_i, $i = 1, 2$ could take the form of any of the pdfs discussed above. The cdf corresponding to (29) is:

$$P(x) = \alpha P_1(x) + (1 - \alpha) P_2(x),$$

where P_i is the cdf corresponding to p_i. Multiplicative twofold mixtures can be defined by the cdfs

$$P(x) = P_1(x) P_2(x)$$

and

$$P(x) = 1 - \{1 - \alpha P_1(x)\}\{1 - \beta P_2(x)\}$$

for $0 < \alpha < 1$ and $0 < \beta < 1$. Mixtures of more than two components can be defined in the obvious manner.

16. Brittle vs Ductile strength

Suppose $X_1, X_2, \dots, X_n$ are independent and identically distributed random variables, representing the strength of n items. If the distribution of max ($X_1, X_2, \dots, X_n$) belongs to the same distribution family of $X_1, X_2, \dots, X_n$ then the distribution will be an appropriate model for ductile strength. On the other hand, if the distribution of min ($X_1, X_2, \dots, X_n$) belongs to the same distribution family of $X_1, X_2, \dots, X_n$ then the distribution will be an appropriate model for brittle strength. Of the various Weibull distributions discussed in Sections 1 to 15,

- The following will be appropriate models for brittle strength: the standardWeibull distribution (Eq. (2)); the log Weibull distribution (Eq. (8)); Xie et al.'s Weibull distribution (Eq. (19)); Lai et al.'s Weibull distribution (Eq. (22)); the generalized Weibull distributions (Eqs. (24), (25) and (26)); the four and five parameterWeibull distributions (Eqs. (27) and (28)).
- The following will be appropriate models for ductile strength: the reflected Weibull distribution (Eq. (4)); the inverse Weibull distribution (Eq. (10)); the exponentiated Weibull distribution (Eq. (17)).

The doubleWeibull distribution given by (6) can be used to model both brittle and ductile strengths.

17. Comparison study

In this section, we provide a comparison of the various Weibull distributions discussed in Sections 1 to 15. We use fracture toughness data from the six different materials: $Bi_2Sr_2CaCu_2O_{8+x}$, Alumina (Al_2O_3), Silicon Nitride (Si_3N_4), Sialon ($Si_{6-x}Al_xO_xN_{8-x}$), Pyroceram 9606, and Titanium Diboride (TiB2). These data are taken from the web–site: http://www.ceramics.nist.gov/srd/summary/ftmain.htm.

For the interest of the readers, we have reproduced the data in Table 1. Some summary statistics of the data are given in Table 2. Note that $Bi_2Sr_2CaCu_2O_{8+x}$ and Sialon ($Si_{6-x}Al_xO_xN_{8-x}$) are two of the toughest materials while Titanium Diboride (TiB_2) is the lightest material.

We fitted eight of theWeibull distributions to each of the six data sets: the standardWeibull distribution (Eq. (1)), inverse Weibull distribution (Eq. (9)), Marshall and Olkin's Weibull distribution (Eq. (13)), pseudo Weibull distribution (Eq. (14)), Stacy's Weibull distribution (Eq. (15)), exponentiated Weibull

Table 1. Fracture toughness data for six different materials.

Material	Fracture toughness data (in the units of MPa $m^{1/2}$)
Material 1 Titanium Diboride (TiB_2)	3.7,5.75,4.25,6.4,4.87,2.3,5.14,4.6,6,5.2, 5.36
Material 2 Pyroceram 9606	2.5,3.17,2.69,2.14,2.07,2.8,2.5,2.25
Material 3 Sialon ($Si_{6-x}Al_xO_xN_{8-x}$)	3.05,2.9,2.75,2.7,2.65,3.15,3.75,3.8,3.72, 3.52,3.44,3.26,2.99,2.79,3,3.18,3.66,3.2, 3.3,3.5,3.1,4.65,3.42,3.38,3.29
Material 4 Silicon Nitride (Si_3N_4)	8.3,7.2,3.2,4.96,7.81,6.59,4.9,4.1,4.5,4.7, 3.12,2.7,6.75
Material 5 Alumina (Al_2O_3)	5.5,5,4.9,6.4,5.1,5.2,5.2,5,4.7,4,4.5,4.2, 4.1,4.56,5.01,4.7,3.13,3.12,2.68,2.77,2.7, 2.36,4.38,5.73,4.35,6.81,1.91,2.66,2.61, 1.68,2.04,2.08,2.13,3.8,3.73,3.71,3.28, 3.9,4,3.8,4.1,3.9,4.05,4,3.95,4,4.5,4.5, 4.2,4.55,4.65,4.1,4.25,4.3,4.5,4.7,5.15, 4.3,4.5,4.9,5,5.35,5.15,5.25,5.8,5.85,5.9, 5.75,6.25,6.05,5.9,3.6,4.1,4.5,5.3,4.85, 5.3,5.45,5.1,5.3,5.2,5.3,5.25,4.75,4.5,4.2, 4,4.15,4.25,4.3,3.75,3.95,3.51,4.13,5.4,5, 2.1,4.6,3.2,2.5,4.1,3.5,3.2,3.3,4.6,4.3,4.3, 4.5,5.5,4.6,4.9,4.3,3,3.4,3.7,4.4,4.9,4.9,5
Material 6 $Bi_2Sr_2CaCu_2O_{8+x}$	3.2,3.9,2.7,3.2,1.9,1.2,1.8,1.4,1.8,2.9,2.8, 2.4

Table 2. Summary statistics of fracture toughness data.

	Material 1	Material 2	Material 3	Material 4	Material 5	Material 6
Mean	4.870	2.515	3.286	5.295	4.325	2.433
Median	5.140	2.500	3.260	4.900	4.380	0.822
Minimum	2.300	2.070	2.650	2.700	1.680	1.200
Maximum	6.400	3.170	4.650	8.300	6.810	3.900
Standard deviation	1.151	0.368	0.435	1.855	1.018	0.822

distribution (Eq. (16)), Xie et al.'s Weibull distribution (Eq. (18)) and the Lai et al.'s Weibull distribution (Eq. (21)). The fitting of the distributions was performed by the method of maximum likelihood. For example, for fitting the standard Weibull distribution given by (1) we maximized

$$L(\alpha, \lambda) = \alpha^n \lambda^{n\alpha} \left(\prod_{i=1}^{n} x_i \right)^{\alpha-1} \exp\left(-\lambda \sum_{i=1}^{n} x_i^\alpha \right)$$

or equivalently

$$\log L(\alpha, \lambda) = n \log \alpha + n\alpha \log \lambda + (\alpha - 1) \sum_{i=1}^{n} \log x_i - \lambda \sum_{i=1}^{n} x_i^\alpha$$

(where x_i, i = 1, 2, . . . , n are the observed fracture toughness) with respect to the two parameters α

Table 3. Fitted models and estimates.

Model	Parameter estimates	NLLH
Material 1: Titanium Diboride (TiB_2)		
Standard Weibull	$\hat{\alpha}$ =5.626 and $\hat{\lambda}$ =0.189	16.076
Inverse Weibull	$\hat{\alpha}$ =2.920 and $\hat{\lambda}$ =4.047	21.182
Marshall & Olkin's Weibull	$\hat{\alpha}$ =4.596, $\hat{\lambda}$ =0.208 and $\hat{\nu}$ =2.800	24.017
Pseudo Weibull	$\hat{\alpha}$ =5.010 and $\hat{\lambda}$ =0.198	16.202
Stacy's Weibull	$\hat{\alpha}$ =0.231, $\hat{\beta}$ =6.234 and $\hat{c}$ =16.394	15.553
Exponentiated Weibull	$\hat{\alpha}$ =15.674, $\hat{\beta}$ =0.164 and $\hat{a}$ =0.241	15.542
Xie et al.'s Weibull	$\hat{\alpha}$ =0.643, $\hat{\beta}$ =0.909 and $\hat{\lambda}$ =0.002	15.660
Lai et al.'s Weibull	$\hat{\beta}$ =0.693, $\hat{\lambda}$ =0.808 and $\hat{\nu}$ =3.688x10^{-6}	38.081
Material 2: Pyroceram 9606		
Standard Weibull	$\hat{\alpha}$ =7.584 and $\hat{\lambda}$ =0.375	3.269
Inverse Weibull	$\hat{\alpha}$ =8.673 and $\hat{\lambda}$ =2.334	2.568
Marshall & Olkin's Weibull	$\hat{\alpha}$ =5.803, $\hat{\lambda}$ =0.408 and $\hat{\nu}$ =2.849	9.324
Pseudo Weibull	$\hat{\alpha}$ =7.023 and $\hat{\lambda}$ =0.384	3.213
Stacy's Weibull	$\hat{\alpha}$ =97.341, $\hat{\beta}$ =0.005 and $\hat{c}$ =0.742	2.664
Exponentiated Weibull	$\hat{\alpha}$ =0.989, $\hat{\beta}$ =3.675 and $\hat{a}$ =4.638x10^{3}	2.516
Xie et al.'s Weibull	$\hat{\alpha}$ =0.076, $\hat{\beta}$ =0.677 and $\hat{\lambda}$ =0.000	3.571
Lai et al.'s Weibull	$\hat{\beta}$ =1.582, $\hat{\lambda}$ =0.001 and $\hat{\nu}$ =2.231	3.619
Material 3: Sialon ($Si_{6-x}Al_xO_xN_{8-x}$)		
Standard Weibull	$\hat{\alpha}$ =6.000 and $\hat{\lambda}$ =0.287	17.358
Inverse Weibull	$\hat{\alpha}$ =9.284 and $\hat{\lambda}$ =3.073	12.568
Marshall & Olkin's Weibull	$\hat{\alpha}$ =5.049, $\hat{\lambda}$ =0.329 and $\hat{\nu}$ =3.744	35.321
Pseudo Weibull	$\hat{\alpha}$ =6.517 and $\hat{\lambda}$ =0.295	16.953
Stacy's Weibull	$\hat{\alpha}$ =108.272, $\hat{\beta}$ =0.007 and $\hat{c}$ =0.761	13.158
Exponentiated Weibull	$\hat{\alpha}$ =1.206, $\hat{\beta}$ =1.720 and $\hat{a}$ =1839.866	12.248
Xie et al.'s Weibull	$\hat{\alpha}$ =0.052, $\hat{\beta}$ =0.582 and $\hat{\lambda}$ =0.000	18.996
Lai et al.'s Weibull	$\hat{\beta}$ =7.433, $\hat{\lambda}$ =9.221x10^{-5} and $\hat{\nu}$ =1.927x10^{-6}	17.466
Material 4: Silicon Nitride (Si_3N_4)		
Standard Weibull	$\hat{\alpha}$ =3.303 and $\hat{\lambda}$ =0.169	25.735
Inverse Weibull	$\hat{\alpha}$ =2.990 and $\hat{\lambda}$ =4.172	26.586
Marshall & Olkin's Weibull	$\hat{\alpha}$ =2.626, $\hat{\lambda}$ =0.198 and $\hat{\nu}$ =2.618	35.625
Pseudo Weibull	$\hat{\alpha}$ =2.725 and $\hat{\lambda}$ =0.194	25.702
Stacy's Weibull	$\hat{\alpha}$ =4.008, $\hat{\beta}$ =2.144 and $\hat{c}$ =1.491	25.660
Exponentiated Weibull	$\hat{\alpha}$ =2.099, $\hat{\beta}$ =0.225 and $\hat{a}$ =2.341	25.692
Xie et al.'s Weibull	$\hat{\alpha}$ =0.001, $\hat{\beta}$ =0.282 and $\hat{\lambda}$ =0.007	25.917
Lai et al.'s Weibull	$\hat{\beta}$ =0.731, $\hat{\lambda}$ =0.859 and $\hat{\nu}$ =1.332x10^{-6}	49.019
Material 5: Alumina (Al_2O_3)		
Standard Weibull	$\hat{\alpha}$ =4.965 and $\hat{\lambda}$ =0.212	168.707
Inverse Weibull	$\hat{\alpha}$ =3.024 and $\hat{\lambda}$ =3.612	210.898
Marshall & Olkin's Weibull	$\hat{\alpha}$ =3.998, $\hat{\lambda}$ =0.238 and $\hat{\nu}$ =2.808	255.848
Pseudo Weibull	$\hat{\alpha}$ =4.390 and $\hat{\lambda}$ =0.225	169.073
Stacy's Weibull	$\hat{\alpha}$ =0.801, $\hat{\beta}$ =4.959 and $\hat{c}$ =5.690	168.556
Exponentiated Weibull	$\hat{\alpha}$ =5.504, $\hat{\beta}$ =0.206 and $\hat{a}$ =0.837	168.581
Xie et al.'s Weibull	$\hat{\alpha}$ =0.007, $\hat{\beta}$ =0.393 and $\hat{\lambda}$ =0.000	168.877
Lai et al.'s Weibull	$\hat{\beta}$ =4.058, $\hat{\lambda}$ =0.001 and $\hat{\nu}$ =0.200	168.576
Material 6: $Bi_2Sr_2CaCu_2O_{8+x}$		
Standard Weibull	$\hat{\alpha}$ =3.462 and $\hat{\lambda}$ =0.369	13.977
Inverse Weibull	$\hat{\alpha}$ =2.899 and $\hat{\lambda}$ =1.922	15.435
Marshall & Olkin's Weibull	$\hat{\alpha}$ =2.795, $\hat{\lambda}$ =0.428 and $\hat{\nu}$ =2.630	23.018
Pseudo Weibull	$\hat{\alpha}$ =2.883 and $\hat{\lambda}$ =0.418	13.974
Stacy's Weibull	$\hat{\alpha}$ =1.230, $\hat{\beta}$ =2.497 and $\hat{c}$ =3.047	13.973
Exponentiated Weibull	$\hat{\alpha}$ =3.347, $\hat{\beta}$ =0.374 and $\hat{a}$ =1.059	13.977
Xie et al.'s Weibull	$\hat{\alpha}$ =0.001, $\hat{\beta}$ =0.305 and $\hat{\lambda}$ =0.010	14.117
Lai et al.'s Weibull	$\hat{\beta}$ =3.487, $\hat{\lambda}$ =0.031 and $\hat{\nu}$ =9.727x10^{-7}	12.978

Table 4. Fitted models and goodness of fit.

Model	Goodness of fit measure					
	Material 1	Material 2	Material 3	Material 4	Material 5	Material 6
Standard Weibull	0.345	0.406	1.500	0.686	2.432	0.520
Inverse Weibull	1.234	0.395	0.852	0.752	11.103	0.748
Marshall & Olkin's Weibull	0.279	0.509	1.541	0.765	3.873	0.534
Pseudo Weibull	0.359	0.396	1.416	0.667	2.538	0.548
Stacy's Weibull	0.282	0.358	0.452	0.621	2.401	0.540
Exponentiated Weibull	0.280	0.381	0.570	0.628	2.408	0.527
Xie et al.'s Weibull	0.259	0.446	1.799	0.713	2.392	0.483
Lai et al.'s Weibull	4.415	0.441	1.445	5.568	2.379	0.530

and λ. The quasi-Newton algorithm nlm in the R software package (Dennis and Schnabel, [24]; Schnabel et al., [25]; Ihaka and Gentleman, [26]) was used to maximize the likelihood. A sample program in R for fitting (1) is illustrated below. Similar programs for fitting the other Weibull distributions can be obtained by contacting the first author.

```
#fitting of the standard Weibull model given by (1)
for data contain in x f<-function (p)
{alpha<-p [1]
lambda<-p [2]
tt<-10000000
if (alpha>0&lambda>0) tt<--n*log (alpha)-n*alpha*
log (lambda)+ (1-alpha)*sum (log (x))
if (alpha>0&lambda>0) tt<-tt+ (lambda**alpha)*
sum (x**alpha)
return (tt)}
est<-nlm (f,p=c (1,1),iterlim=1000)
#the maximum likelihood estimates of alpha &
lambda returned
alpha<-est$estimate [1]
lambda<-est$estimate [2]
```

The maximum likelihood estimates of the parameters as well as the negative logarithm of the maximized likelihood (NLLH) are given in Table 3 for the eight distributions fitted. The goodness of the fitted models was checked by means of probability plots. A probability plot consists of plots of the observed probabilities against the probabilities predicted by the fitted model. For example, for the standard Weibull model, $1-\exp\{-(\hat{\lambda}x_{(i)})^{\hat{\alpha}}\}$ was plotted versus $(i-0.375)/(n+0.25)$, $i = 1, 2, \ldots, n$ (as recommended by Blom [27] and Chambers et al. [28]), where $x_{(i)}$ are the sorted values of the observed fracture toughness. For the inverse Weibull model, $\exp\{-(x_{(i)}/\hat{\lambda})^{-\hat{\alpha}}\}$ was plotted versus $(i-0.375)/(n+0.25)$, $i = 1, 2, \ldots, n$. An objective measure of the goodness of fit was calculated by taking the sum of the absolute differences between the observed probabilities and the probabilities predicted by the fitted model. The values of this measure are shown in Table 4.

Note that five of the fitted distributions have three parameters while the other three distributions have two parameters. Although not all of the distributions are nested, those with the same number of parameters can be compared by means of the standard likelihood ratio test. It follows that the best fits are exhibited by the bottom four models: Stacy's Weibull distribution, exponentiated Weibull distribution, Xie et al.'s Weibull distribution and Lai et al.'s Weibull distribution. This is understandable because these distributions have more parameters, more recently developed and more flexible than the other Weibull distributions. There is no evidence to suggest that certain models perform better for the toughest materials or the lightest materials (see Table 2). It is just that the more flexible models perform better irrespective of the strength of the materials. These observations are confirmed by the values of the goodness of fit measure in Table 4.

18. Conclusions

We have reviewed over 20 known variations of the Weibull distribution which could be used as possible models for material strength. We have discussed the appropriateness of these distributions as models for brittle versus ductile strength. We have also provided a comparison study based on six data sets on fracture toughness. We believe that this review will serve as an important reference, help to provide better modeling and encourage developments of further variations of the Weibull distribution.

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